Fuzzy Sets in Topological Spaces

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***Abstract*—**In topological spaces, intuitionistic fuzzy open set's notions are introduced and studied in this paper. In intuitionistic fuzzy topological spaces, studied some of its properties also.

***Keywords*—**Intuitionistic Fuzzy Open Sets; Intuitionistic Fuzzy Closed Sets; Intuitionistic Fuzzy Topology.

***Abbreviations*—**Fuzzy Set (FS); Intuitionistic Fuzzy Closed Set (IFCS); Intuitionistic Fuzzy Open Set (IFOS); Intuitionistic Fuzzy Topological Spaces (IFTS); Intuitionistic Fuzzy Topology (IFT).

I. INTRODUCTION

SPACES and shapes mathematical study corresponds to topology. Out of set theory and geometry, topology is developed as a field of study, where transformation, dimension and space concepts are analyzed. Generalization closed sets study us initiated by Levine [10] in 1970. In recent years, various topologists have studied notion extensively. This is because, in closed set's generalization, they play a major role and they suggested some new separation axioms. In quantum physics, digital topology and computer science, most of these axioms are utilized. In 1965, Zadeh [9] introduced FS concepts and in 1968, Chang [2] introduced a fuzzy topology. In fuzzy topological spaces, generalized closed sets concepts are introduced by Sundaram and Balasubramanian [3] in 1997.

In 1986, IFS concepts are proposed by Atanassov [1], where fuzzy set is added with non-membership degree using fuzzy sets notion. Intuitionistic fuzzy closed sets, open sets and topological spaces concepts are introduced by Coker [3] in 1997. In topological spaces, intuitionistic fuzzy open set's notions are introduced and studied in this paper. In intuitionistic fuzzy topological spaces, studied some of its properties also.

II. PRELIMINARIES

In this paper, ITFS is represented as X or (X, τ). For X's subset A, closure is represented as cl(A), interior is represented as int(A), A's complement is represented as A^C. In sequel, some basic definitions which are utilized are recalled here.

2.1. Definition 2.1.

Assume A, B are IFSs and are given by B = {<x, $\mu_B(x)$, $\nu_B(x)$ > / x ∈ X}, A ={<x, μ_A (x), ν_A (x) > / x ∈ X,. Then [1]

1. for every x ∈ X, if and only if $\nu_A(x) \geq \nu_B(x)$ and $\mu_A(x) \leq \mu_B(x)$, A ⊆ B
2. If B ⊆ A, A ⊆ B then A = B
3. A^C = {<x, $\mu_A(x)$, $\nu_A(x)$, > / x ∈ X},
4. A ∩ B = {<x, $\nu_A(x) \vee \nu_B(x)$, $\mu_A(x) \wedge \mu_B(x)$, > / x ∈ X},
5. A ∪ B = {<x, $\nu_A(x) \wedge \nu_B(x)$, $\mu_A(x) \vee \mu_B(x)$, > / x ∈ X}.

A = { ⟨x, μA(x), νA(x)⟩/x ∈ X } is replaced by notation A = ⟨x, μA, νA⟩, A = ⟨x, (A/μA, B/μB), (A/νA, B/νB)⟩ is replaced by notation A = ⟨x, (μA, μB), (νA, νB)⟩ for simplification. In X, empty set is given by IFS 0~ = {<x, 0, 1> / x ∈ X}, whole set is given by 1~ = {<x, 1, 0> / x ∈ X}.

2.2. Definition 2.2.

In an IFTS (X, τ), IFS A = <x, μ_A, ν_A> is termed as [4]

1. If int(cl(A)) ⊆ A, then it is termed as intuitionistic fuzzy semi-closed set (IFSCS)
2. If A ⊆ cl(int(A)), then it is termed as intuitionistic fuzzy semi-open set (IFSOS)
3. If cl(int(cl(A))) ⊆ A, then it is termed as intuitionistic fuzzy α -closed set (IF α CS)
4. If A ⊆ int(cl(int(A))), then it is termed as intuitionistic fuzzy α -open set (IF α OS)
5. If cl(int(A)) = A, then it is termed as intuitionistic fuzzy regular closed set (IFRCS)
6. If A = int(cl(A)). Then it is termed as intuitionistic fuzzy regular open set (IFROS)
7. If cl(A) ⊆ U, then it is termed as intuitionistic fuzzy generalized closed set (IFGCS), whenever A⊆U, in X, U is IFOS. IFGCS complement is IFGOS [9],
8. If cl(A) ⊆ U, then it is termed as intuitionistic fuzzy regular generalized closed set (IFRGCS) whenever A ⊆ U, in X, U is IFROS. IFRGCS complement is IFRGOS [7];[8],
9. If αcl(A) ⊆ U, then it is termed as intuitionistic fuzzy generalized α -closed set (IFG α CS), whenever A ⊆ U, in X, U is IF α OS. IFG α CS complement is IFG α OS [6],

10. If αcl(A) ⊆ U, then it is termed as intuitionistic fuzzy α -generalized closed set (IFαGCS) [6] whenever A ⊆ U, in X, U is IFOS.

11. If cl(A) ⊆ U, then it is termed as intuitionistic fuzzy W-closed set (IFWCS), whenever A ⊆ U, in X, U is IFSOS. IFWCS complement is IFWOS [5],

2.3. Definition 2.3.

Assume non-empty set as X [1]. In X, IFS A is an object with A = {<x, $\mu_A(x)$, $\nu_A(x)$> / x ∈ X} form, where, every element x ∈ X in set A, membership degree is represented using function $\mu_A : X \rightarrow [0, 1]$ and it is named as μ_A (x), non-membership degree is represented as $\nu_A : X \rightarrow [0, 1]$ and it is named as ν_A (x), for every x ∈ X, $0 \leq \mu_A (x)+\nu_A (x) \leq 1$. In X, all IFS is represented as IFS (X).

2.4. Definition 2.4.

In X, IFSs family τ is an IFT. Following axioms are satisfied by this [3].

1. 0~, 1~ ∈ τ,
2. G1 ∩ G2 ∈ τ for any G1, G2 ∈ τ,
3. ∪Gi ∈ τ for any family {Gi / i ∈ J} ⊆ τ.

In X, pair (X, τ) is termed as IFTS and in τ, any IFS is termed as IFOS. In an IFTS (X, τ), IFOS A's A^C complement is termed as IFCS in X.

2.5. Definition 2.5.

In X, assume (X, τ) as IFTS and A=<x, μ_A, ν_A> as IFS. Then [3]

1. int(A) = ∪{ G ⊆ A, G / G is an IFOS in X },
2. cl(A) = ∩{ A ⊆ K and K / K is an IFCS in X },
3. $cl(A^C) = (int(A))^C$,
4. $int(A^C) = (cl(A))^C$.

Result 2.5.
In IFTS (X, τ), any two IFS are assumed as A and B. Then cl(A ∪ B)=cl(A) ∪ cl(B) [5].

III. INTUITIONISTIC FUZZY g̈ OPEN SETS

3.1. Definition 3.1.

In X, if Ac is IFg̈ CS, IFS A is termed as IFg̈ OS.

Theorem 3.2.
For any IFTS (X, τ), we have following:

1. Each IFOS is IFg̈ OS,
2. Each IFROS is IFg̈ OS.

Proof: Obvious.

Remark 3.3.
In general, converses of above theorem need not be true as seen from following examples.

Example 3.4.
Assume X = {a, b} and G = <x, (0.5, 0.6), (0.5, 0.4)>. Then on X, τ = {0~, G, 1~} is IFT, IFS A = <x, (0.6, 0.6), (0.4, 0.4)> is IFg̈ OS but in (X, τ), it is not IFOS and IFROS.

3.5. Theorem 3.5.

An IFS A of IFTS (X, τ) is IFg̈ OS if and only if U ⊆ int(A) whenever U ⊆ A, U is IFSGCS.

Proof:
In X, assume that A is IFg̈ OS. Assume U as IFSGCS so that U ⊆ A. Then U c is IFSGOS, Ac ⊆ U c. Then in X, Ac is IFg̈ CS as per assumption. Therefore cl(Ac) ⊆ U c. Hence U ⊆ int(A). Conversely, assume U as IFSGOS in X so that Ac ⊆ U. Then U c ⊆ A, U c is IFSGCS. Therefore U c ⊆ int(A). Since U c ⊆ int(A), (int(A))c ⊆ U that is cl(Ac) ⊆ U. Thus Ac is IFg̈ CS. Hence in X, A is IFg̈ OS.

3.6. Theorem 3.6.

If A is IFg̈ OS, int(A) ⊆ B ⊆ A, then B is IFg̈ OS.

Proof:
If int(A) ⊆ B ⊆ A, then Ac ⊆ Bc ⊆ (int(A))c = cl(Ac). Since Ac is IFg̈ CS then Bc is IFg̈ CS. Therefore B is IFg̈OS.

IV. INTUITIONISTIC FUZZY g̈ CLOSED SETS

4.1. Definition 4.1.

In IFTS (X, τ), if cl(A) ⊆ U, then IFS A is termed as intuitionistic fuzzy g̈ closed set(IFg̈CS) whenever A ⊆ U, in (X, τ), U is an IFSGOS.
An IFTS (X, τ)'s all IFg̈CSs family is represented as IFg̈CS (X). In (X, τ), IFg̈CS complement is IFg̈OS.

4.2. Theorem 4.2.

In (X, τ), each IFCS is IFg̈CS, but not conversely.

Proof:
Assume A as IFCS, A ⊆ U where in (X, τ), U is IFSGOS. Then cl(A) = A ⊆ U as per hypothesis. Hence in (X, τ), A is IF g̈CS.

Example 4.3.
Assume X = {a, b}, G = <x, (0.5, 0.6), (0.5, 0.4)>. Then, on X, τ ={0~, G, 1~} is IFT, IFS A = <x, (0.4, 0.4), (0.6, 0.6) > is IFg̈ CS, but in (X, τ), it is not IFCS.

4.4. Theorem 4.4.

In (X, τ), each IFRCS is IFg̈CS, but not conversely.

Proof:
Since each IFRCS is IFCS and using Theorem 3.3., in X, A is IFg̈CS.

Example 4.5.
Assume X = {a, b}, G = <x, (0.5, 0.6), (0.5, 0.4)>. Then, on X, τ ={0~, G, 1~} is IFT, IFS A = <x, (0.4, 0.4), (0.6, 0.6)> is IFg̈ CS but in (X, τ), it is not an IFRCS.

4.6. Theorem 4.6.

In (X, τ), each IFg̈CS is IFGCS, but not conversely.

Proof:
Assume A ⊆ U, U as IFOS in (X, τ). Since each IFOS is IFSGOS and cl(A) ⊆ U as per hypothesis. Hence in (X, τ), A is IFGCS.

Example 4.7.
Assume X = {a, b}, G = <x, (0.8, 0.8), (0.2, 0.1)>. Then, on X, τ ={0~, G, 1~} is IFT and IFS A = <x, (0.8, 0.7), (0.1, 0.3)> is IFGCS but in (X, τ), it is not an IFg̈CS.

4.8. Theorem 4.8.

In (X, τ), each IF$\ddot{g}$ CS is IFRGCS, but not conversely.

Proof:

Assume A ⊆ U, in (X, τ), U as IFROS. Since each IFROS is IFSGOS, cl(A) ⊆ U, as per hypothesis,. Hence A is an IFRGCS in (X, τ).

Example 4.9.

Assume X = {a, b}, G = <x, (0.8, 0.8), (0.2, 0.1)>. Then on X, τ ={0~, G, 1~} is IFT, IFS A = <x, (0.9, 0.7), (0.1, 0.3)> is IFRGCS but in (X, τ), it is not IF$\ddot{g}$CS.

4.10. Theorem 4.10.

In (X, τ), each IF$\ddot{g}$CS is IFG α CS, but not conversely.

Proof:

Assume A ⊆ U, in (X, τ), U as IF α OS. Since each IF α OS is IFSGOS and αcl(A) ⊆ cl(A) ⊆ U as per hypothesis. Hence, in (X, τ), A is IFG α CS

Example 4.11.

Assume X = {a, b}, G = <x, (0.3, 0.2), (0.7, 0.7)>. Then, on X, τ ={0~, G, 1~} is IFT, IFS A = <x, (0.6, 0.6), (0.3, 0.4)> is IFG α CS but in (X, τ), it is not an IF$\ddot{g}$ CS.

4.12. Theorem 4.12.

In (X, τ), each IF$\ddot{g}$ CS is IFWCS, but not conversely.

Proof:

Assume A ⊆ U, in (X, τ), U as IFSOS. Since each IFSOS is IFSGOS and cl(A) ⊆ U as per hypothesis. Hence, in (X, τ), A is IFWCS.

Example 4.13.

Assume X = {a, b}, G = <x, (0.7, 0.7), (0.3, 0.3)>. Then, on X, τ ={0~, G, 1~} is IFT, IFS A = <x, (0.4, 0.4), (0.6, 0.6)> is an IFWCS but in (X, τ), it is not an IF$\ddot{g}$ CS.

Relationship between different IFCS types with IF$\ddot{g}$ CS is provided in the below mentioned diagram.

↗ IFWCS
IFRCS → IFCS → IF$\ddot{g}$ CS → IFGCS → IFRGCS
↘ IFGαCS

In this "A → B" means A implies B but not conversely.

4.14. Theorem 4.14.

Assume A, B as two IF$\ddot{g}$ CSs in IFTS (X, τ) then in X, A ∪ B is IF$\ddot{g}$ CS.

Proof:

Assume U as IFSGOS in (X, τ) so that A ∪ B ⊆ U. Since A, B are IF g¨ CSs, cl(A) ⊆ U, cl(B) ⊆ U. Therefore using Result 1.2.6, cl(A) ∪ cl(B) = cl(A ∪ B) ⊆ U. Hence, in (X, τ), A ∪ B is IF g¨ CS.

4.15. Theorem 4.15.

If A is IF$\ddot{g}$ CS and A ⊆ B ⊆ cl(A), then B is IF$\ddot{g}$ CS.

Proof:

Assume U as IFSGOS so that B ⊆ U. Since A is IF$\ddot{g}$ CS, cl(A) ⊆ U. cl(B) ⊆ cl(A) ⊆ U as per hypothesis. Hence B is IF$\ddot{g}$ CS

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