

A Common Fixed Point Theorem for Self Maps on a Metric Space using Pseudo Altering Distance Function and a Deficit Function

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Abstract—A generalized weak contraction condition involves an altering distance function which is non-decreasing and continuous. However, in dealing with problems, the altering distance function fails to be continuous throughout. There is a need to develop tools to handle such situations. Further, in many of the common fixed point results while dealing with generalized contractions involving an altering distance function ψ and a deficit function ϕ , the absence of any link between ψ and ϕ may not give good results. Further, a suitable link between ψ and ϕ will lead to fruitful results. Keeping these things in view, we show a way to successfully avoid continuity of the altering distance function. Improving a suitable link between ψ and ϕ we prove common fixed point results, from which the results of earlier authors follow as corollaries.

Keywords—Altering Distance, Deficit Function, Fixed Point, Generalized Weak Contraction, Metric Space, Pseudo Altering Distance, Weakly Compatible Maps, Weak Contraction

Mathematics Subject Classifications—47H10, 54H25

I. INTRODUCTION

THE study of fixed points of mappings satisfying certain conditions has been at the centre of rigorous research activity. Rhoades (1977) showed that there are several possible types of extended forms of contraction pairs. Jungck (1986) introduced the notion of compatible mappings which are more general than commuting and weakly commuting mappings. In 1998, Jungck & Rhoades (1998) introduced the concept of weakly compatible maps and showed that compatible maps are weakly compatible but not conversely. Chugh & Kumar (2001) proved a fixed point theorem for weakly compatible maps without appeal to continuity.

Khan et al., (1998) introduced the notion of altering distance and used it to solve fixed point problems in metric spaces. Recently many authors, for example, [Sastry & Babu, 1999; Naidu, 2003; Singh & Dimri, 2011] used the altering distance function and obtained some fixed point theorems. Further, the concept of weak contraction was introduced in 1997 by Ya I. Alber & Guerre (1997) in Hilbert spaces and subsequently extended to metric spaces by Rhoades (2001).

Recently, Popescu (2011) proved fixed point problem involving weak contraction and mapping satisfying weak contractive type inequalities.

Gairola & Ram Krishan (2013) presented fixed point results for three self maps satisfying a generalized weak contraction condition by using the concept of weakly compatible maps in a complete metric space. These results extend and generalized the results of Choudhary et al., (2011).

A generalized weak contraction condition involves an altering distance function which is increasing and continuous. In this paper, we introduce the notion of a pseudo altering distance function (doing away with continuity) and obtain a common fixed point theorem.

In the results of earlier authors, dealing with generalized contractions involving an altering distance function ψ and a deficit function ϕ no link is given between ψ and ϕ . In this paper, besides introducing a pseudo altering distance function ψ , we also insist on a link between ψ and the deficit function ϕ . We observe that, when ψ is continuous, this link is satisfied automatically. With the aid of a pseudo altering distance function ψ and a link between ψ and the deficit

function φ , we prove a common fixed point theorem which extends the results of Gairola & Ram Krishan (2013).

II. PRELIMINARIES

In this section we state some known definitions which we use in the next section.

Definition 1.1: [Rhoades, 2001]

A mapping $T: X \rightarrow X$, where (X, d) is a metric space, is said to be weakly contractive, for $x, y \in X$, $d(Tx, Ty) \leq d(x, y) - \varphi(d(x, y))$, where,

$\varphi: [0, \infty) \rightarrow [0, \infty)$ is a continuous non - decreasing function such that $\varphi(t) = 0$ if and only if $t = 0$. If one takes $\varphi(t) = (1 - k)t$, where $0 < k < 1$, a weak contraction reduces to a Banach contraction.

Definition 1.2: [Khan et al., 1998]

A function $\psi: [0, \infty) \rightarrow [0, \infty)$ is called an altering distance function, if the following properties are satisfied,

(1.2.1): ψ is monotonically increasing and continuous,

(1.2.2): $\psi(t) = 0$ if and only if $t = 0$.

Definition 1.3: [Choudhary et al., 2011]

Let (X, d) be a metric space, T a self mapping of X . We shall call T a generalized weakly contractive mapping if for all $x, y \in X$.

$$\psi(d(Tx, Ty)) \leq \psi(m(x, y) - \varphi(\max\{d(x, y), d(x, Tx), d(y, Ty)\}))$$

where,

$$m(x, y) = \max\{d(x, y), d(x, Tx), d(y, Ty), \frac{1}{2}[d(x, Ty) + d(y, Tx)]\}$$

ψ is an altering distance function and

$\varphi: [0, \infty) \rightarrow [0, \infty)$ is a continuous function with $\varphi(t) = 0$ if and only if $t = 0$.

Definition 1.4:

Functions $\varphi: [0, \infty) \rightarrow [0, \infty)$, such that φ is a continuous with $\varphi(t) = 0$ if and only if $t = 0$ are called deficit functions.

A generalized weakly contractive mapping is more general than that satisfying,

$d(Tx, Ty) \leq km(x, y)$, for some $0 \leq k < 1$ and is included in those mappings which satisfy $d(Tx, Ty) < m(x, y)$.

Definition 1.5: [Jungck, 1986]

Let S and T be mappings from a metric space (X, d) into itself. The mappings S and T are said to be compatible if, $\lim_{n \rightarrow \infty} d(STx_n, TSx_n) = 0$, whenever $\{x_n\}$ is a sequence in X such that $\lim_{n \rightarrow \infty} Sx_n = t = Tx_n$, for some $t \in X$.

Definition 1.6: [Jungck & Rhoades, 1998]

Let S and T be mappings from a metric space (X, d) into itself. The mappings S and T are said to be weakly compatible if, they commute at their coincidence points, that

is, if $Tu = Su$ for some $u \in X$, then $TSu = STu$. In this connection if we write $v = Tu = Su$ then, we say that v is a point of coincidence of (S, T) .

III. MAIN RESULT

Before we go to our main result, we introduce the notion of a pseudo altering distance function.

Definition 2.1:

Suppose $\psi: [0, \infty) \rightarrow [0, \infty)$ is a non - decreasing function such that,

(2.1.1): $\psi(t) = 0$ if and only if $t = 0$, and

(2.1.2): 0 is a limit point of the range of ψ , that is, 0 is a limit point of $\psi([0, \infty))$.

Then ψ is called a pseudo altering distance function.

Remark 2.2:

We observe that every altering distance function is a pseudo altering distance function. If ψ is an altering distance function, then $\psi(t) = 0$ if and only if $t = 0$.

Hence, for any $\epsilon > 0$, $[0, \epsilon)$ is connected and hence $\psi([0, \epsilon))$ is connected. Consequently $\psi([0, \epsilon))$ is an interval of containing $[0, \delta)$ for some $\delta > 0$ (from (2.1.1)).

Hence 0 is a limit point of the range of ψ .

The following example shows that the notion of a pseudo altering distance is a proper generalization of the notion of altering distance.

Example 2.3:

Define $\psi: [0, \infty) \rightarrow [0, \infty)$ as follows.

$$\psi(t) = \begin{cases} 0 & \text{if } t = 0 \\ \frac{1}{n} & \text{if } t \in \left[\frac{1}{n+1}, \frac{1}{n}\right], n = 1, 2, \dots \\ n & \text{if } t \in (n-1, n], n = 2, 3, \dots \end{cases}$$

Then ψ is a pseudo altering distance function but not an altering distance function (being not continuous).

Now we state and prove our main result.

Theorem 2.4:

Let (X, d) be a complete metric space. Let $S, f, g: X \rightarrow X$ be self mappings such that for all $x, y \in X$, $S(X) \subset f(X) \cap g(X)$,

$$\psi(d(Sx, Sy)) \leq \psi(\max\{d(fx, gy), d(Sx, fx), d(Sy, gy), \frac{1}{2}[d(Sx, gy) + d(Sy, fx)]\}) - \varphi(\max\{d(fx, gy), d(Sx, fx), d(Sy, gy)\}) \rightarrow (2.4.1)$$

where,

(2.4.2): $\varphi: [0, \infty) \rightarrow [0, \infty)$ is a continuous function with $\varphi(t) = 0$ if and only if $t = 0$.

(2.4.3): $\psi: [0, \infty) \rightarrow [0, \infty)$ is a pseudo altering distance function.

(2.4.4): $\psi(\epsilon + 0) - \psi(\epsilon - 0) < \varphi(\epsilon) \forall \epsilon > 0$.

Then (S, f) and (S, g) have a coincidence point.

Further if (S, f) and (S, g) are weakly compatible pairs, then S, f and g have a unique common fixed point.

Proof:

Let $x_0 \in X$. We define a sequence $\{y_n\}$ such that

$$y_{2n} = Sx_{2n} = fx_{2n+1}$$

$$y_{2n+1} = Sx_{2n+1} = gx_{2n+2}, \text{ for all } n \in N \rightarrow (2.4.5)$$

$$\begin{aligned} & \text{Now we using (2.4.1), we have, } \psi(d(y_{2n+1}, y_{2n+2})) = \\ & \psi(d(Sx_{2n+1}, Sx_{2n+2})) \\ & \leq \psi(\max\{d(fx_{2n+1}, gx_{2n+2}), d(Sx_{2n+1}, fx_{2n+1}), \\ & d(Sx_{2n+2}, gx_{2n+2}), \frac{1}{2}[d(Sx_{2n+1}, gx_{2n+2}) + \\ & d(Sx_{2n+2}, fx_{2n+1})]\}) \\ & - \varphi\left(\max\left\{\frac{d(fx_{2n+1}, gx_{2n+2}), d(Sx_{2n+1}, fx_{2n+1}),}{d(Sx_{2n+2}, gx_{2n+2})}\right\}\right) \\ & \leq \psi(\max\{d(y_{2n}, y_{2n+1}), d(y_{2n+1}, y_{2n}), d(y_{2n+2}, y_{2n+1}), \\ & \frac{1}{2}[d(y_{2n+1}, y_{2n+1}) + d(y_{2n+2}, y_{2n})]\}) - \\ & \varphi(\max\{d(y_{2n}, y_{2n+1}), d(y_{2n+1}, y_{2n}), d(y_{2n+2}, y_{2n+1})\}) \\ & = \psi(d(y_{2n+1}, y_{2n+2})) - \varphi(d(y_{2n+1}, y_{2n+2})) \rightarrow (2.4.6) \end{aligned}$$

Suppose,

$$d(y_{2n}, y_{2n+1}) \leq d(y_{2n+1}, y_{2n+2}) \rightarrow (2.4.7)$$

Then for some n from (2.4.5) we get,

$$\begin{aligned} & \psi(d(y_{2n+1}, y_{2n+2})) \\ & \leq \psi(d(y_{2n+1}, y_{2n+2})) - \varphi(d(y_{2n+1}, y_{2n+2})) \\ & \therefore \varphi(d(y_{2n+1}, y_{2n+2})) = 0, \text{ so that } y_{2n+1} = y_{2n+2} \end{aligned}$$

Now from (2.4.6) we get, $y_{2n} = y_{2n+1} = y_{2n+2}$.

Consequently, from (2.4.5),

$$\begin{aligned} y_{2n} &= Sx_{2n} = fx_{2n+1} \\ y_{2n+1} &= Sx_{2n+1} = gx_{2n+2}, \\ y_{2n+2} &= Sx_{2n+2} = fx_{2n+3} \text{ for all } n \in N \end{aligned}$$

Hence,

$$Sx_{2n+1} = y_{2n+1} = y_{2n} = fx_{2n+1}$$

Showing that x_{2n+1} is a coincidence point of S and f ,

and

$$Sx_{2n+2} = y_{2n+2} = y_{2n+1} = gx_{2n+2}$$

Showing that x_{2n+2} is a coincidence point of S and g .

Further y_{2n+1} is a point of coincidence of (S, f) and (S, g) .

$\therefore$ We may suppose that,

$$d(y_{2n}, y_{2n+1}) > d(y_{2n+1}, y_{2n+2}) \text{ for every } n$$

Similarly, we can show that,

$$d(y_{2n+1}, y_{2n+2}) > d(y_{2n+2}, y_{2n+3}) \forall n$$

Thus,

$$d(y_{n+1}, y_{n+2}) > d(y_n, y_{n+1}) \forall n$$

Write $\alpha_n = d(y_n, y_{n+1})$.

Then $\{\alpha_n\}$ is strictly decreasing sequence and hence decreases to a point say, p .

From (2.4.6) we have,

$$\psi(\alpha_{n+1}) \leq \psi(\alpha_n) - \varphi(\alpha_n) < \psi(\alpha_n) \rightarrow (2.4.8).$$

Hence, $\{\psi(\alpha_n)\}$ is a decreasing sequence and hence decreases to, say, q .

$$\therefore \psi(\alpha_n) \geq \psi(p) \forall n \Rightarrow q \geq \psi(p).$$

Letting $n \rightarrow \infty$ in (2.4.8), we get,

$$\left. \begin{array}{l} q \leq q - \varphi(p) \\ (\because \varphi \text{ is continuous}) \end{array} \right\} \Rightarrow \varphi(p) = 0 \Rightarrow p = 0.$$

Thus $\alpha_n \downarrow 0$.

Now we show that $q = 0$.

Suppose $q > 0$. Then from (2.1.2) there exists $t > 0$ such that $0 < \psi(t) < q$.

Then $\psi(t) < q \leq \psi(\alpha_n) \forall n$, so that $t < \alpha_n \forall n$. ($\because \psi$ is non – decreasing)

Hence $t \leq 0$, a contradiction.

Consequently $q = 0$.

Thus $d(y_n, y_{n+1}) \downarrow 0$ and $\psi(d(y_n, y_{n+1})) \downarrow 0$.

Now we shall show that $\{y_n\}$ is a Cauchy sequence.

Suppose that $\{y_n\}$ is not a Cauchy sequence. Then there exist an $\epsilon > 0$ and two sequences $m(k)$ and $n(k)$ such that and

$$\begin{aligned} & m(k) > n(k) > k \\ & d(y_{n(k)}, y_{m(k)}) \geq \epsilon \\ & \text{for every integer } k \end{aligned} \rightarrow (2.4.9).$$

Let $m(k)$ be the least integer exceeding $n(k)$ satisfying (2.4.9) so that,

$$d(y_{n(k)}, y_{m(k)-1}) < \epsilon \rightarrow (2.4.10)$$

Using triangular inequality in (2.4.10), we have,

$$\begin{aligned} \epsilon & \leq d(y_{n(k)}, y_{m(k)}) \\ & \leq d(y_{n(k)}, y_{m(k)-1}) \\ & \quad + d(y_{m(k)-1}, y_{m(k)}) \end{aligned}$$

$$\text{i.e., } \epsilon \leq d(y_{n(k)}, y_{m(k)}) \leq \epsilon + d(y_{m(k)-1}, y_{m(k)})$$

Now letting $k \rightarrow \infty$, we have

$$\lim_{k \rightarrow \infty} d(y_{n(k)}, y_{m(k)}) = \epsilon \rightarrow (2.4.11)$$

Again,

$$\begin{aligned} & d(y_{n(k)}, y_{m(k)}) \leq d(y_{n(k)}, y_{n(k)+1}) \\ & + d(y_{n(k)+1}, y_{m(k)+1}) + d(y_{m(k)+1}, y_{m(k)}) \text{ and} \\ & d(y_{n(k)+1}, y_{m(k)+1}) \leq d(y_{n(k)+1}, y_{n(k)}) \\ & + d(y_{n(k)}, y_{m(k)}) + d(y_{m(k)}, y_{m(k)+1}) \end{aligned}$$

On letting $k \rightarrow \infty$, we have,

$$\lim_{k \rightarrow \infty} d(y_{n(k)+1}, y_{m(k)+1}) = \epsilon \rightarrow (2.4.12)$$

In a similar way, we can show that,

$$\lim_{k \rightarrow \infty} d(y_{n(k)}, y_{m(k)+1}) = \epsilon \rightarrow (2.4.13)$$

We may suppose, without loss of generality, that both $m(k)$ and $n(k)$ are even.

Now for $x = x_{n(k)}, y = x_{m(k)+1}$,

We have from (2.4.1),

$$\begin{aligned} & \psi(d(y_{n(k)+1}, y_{m(k)+2})) = \psi(d(Sx_{n(k)+1}, Sx_{m(k)+2})) \\ & \leq \psi(\max\{d(fx_{n(k)+1}, gx_{m(k)+2}), d(Sx_{n(k)+1}, fx_{n(k)+1}), \\ & d(Sx_{m(k)+2}, gx_{m(k)+2}), \frac{1}{2}[d(Sx_{n(k)+1}, gx_{m(k)+2}) \\ & + d(Sx_{m(k)+2}, fx_{n(k)+1})]\}) \\ & - \varphi\left(\max\left\{\frac{d(fx_{n(k)+1}, gx_{m(k)+2}), d(Sx_{n(k)+1}, fx_{n(k)+1}),}{d(Sx_{m(k)+2}, gx_{m(k)+2})}\right\}\right) \end{aligned}$$

$$\begin{aligned} & \leq \psi \mid \max \frac{1}{2} [d(y_{n(k)+1}, y_{m(k)+1}) + d(y_{m(k)+2}, y_{n(k)})] \\ & - \varphi\left(\max \frac{d(y_{n(k)}, y_{m(k)+1}), d(y_{n(k)+1}, y_{n(k)}),}{d(y_{m(k)+2}, y_{m(k)+1})}\right) \\ & \rightarrow (2.4.14) \end{aligned}$$

Let $\delta > 0$. Then from (2.4.11), (2.4.12) and (2.4.13), there exists N such that for $k \geq N$,

$$d(y_{n(k)}, y_{m(k)+1}) < \epsilon + \delta$$

$$\begin{aligned} d(y_{n(k)+1}, y_{n(k)}) &< \partial \\ d(y_{m(k)+2}, y_{m(k)+1}) &< \partial \\ d(y_{n(k)+1}, y_{m(k)+1}) &< \epsilon + \partial \\ d(y_{m(k)+2}, y_{n(k)+1}) &< \epsilon + \partial \end{aligned}$$

Hence from (2.4.14) and monotonic increasing property of ψ , we get, for $k \geq N$,

$$\begin{aligned} \psi(d(y_{n(k)+1}, y_{m(k)+2})) &\leq \psi(\epsilon + \partial) \\ &- \varphi\left(\max\left\{\begin{aligned} d(y_{n(k)}, y_{m(k)+1}), d(y_{n(k)+1}, y_{n(k)}), \\ d(y_{m(k)+2}, y_{m(k)+1}) \end{aligned}\right\}\right) \end{aligned}$$

Now, letting $k \rightarrow \infty$, since φ is continuous, we get,

$$\lim_{k \rightarrow \infty} \sup \psi(d(y_{n(k)+1}, y_{m(k)+2})) \leq \psi(\epsilon + \partial) - \varphi(\epsilon).$$

$$\therefore \psi(\epsilon - 0) \leq$$

$$\lim_{k \rightarrow \infty} \sup \psi(d(y_{n(k)+1}, y_{m(k)+2})) \leq \psi(\epsilon + \partial) - \varphi(\epsilon).$$

This being true for every $\partial > 0$, we get,

$$\psi(\epsilon - 0) \leq \psi(\epsilon + 0) - \varphi(\epsilon), \text{ so that,}$$

$\varphi(\epsilon) \leq \psi(\epsilon + 0) - \psi(\epsilon - 0) < \varphi(\epsilon)$ a contradiction, from (2.4.4).

Hence $\{y_n\}$ is a Cauchy sequence.

Since X is complete there exists a point z in X such that,

$$\lim_{n \rightarrow \infty} y_{2n} = \lim_{n \rightarrow \infty} Sx_{2n} = \lim_{n \rightarrow \infty} fx_{2n+1} = z \text{ and}$$

$$\lim_{n \rightarrow \infty} y_{2n+1} = \lim_{n \rightarrow \infty} Sx_{2n+1} = \lim_{n \rightarrow \infty} gx_{2n+2} = z.$$

Since $S(X) \subset f(X) \cap g(X)$, there exist points u and $v \in X$ such that $fu = z$ and $gv = z$.

We shall prove that,

$$z = fu = Su \text{ and } z = gv = Sv.$$

Suppose $Su \neq z$,

$$\text{Write } \epsilon = d(Su, z) > 0.$$

Let $0 < \delta < \epsilon$. Then there exists N such that,

$$d(z, y_{2n-1}) < \frac{\epsilon}{2}, d(y_{2n}, y_{2n-1}) < \frac{\epsilon}{2} \text{ and}$$

$$\frac{1}{2} [d(Su, y_{2n-1}) + d(y_{2n}, z)] < \frac{\epsilon}{2} + \frac{\delta}{2} \quad \forall n \geq N$$

$$\therefore \psi(d(Su, fx_{2n+1})) = \psi(d(Su, Sx_{2n}))$$

$$\leq \psi(\max\{d(fu, gx_{2n}), d(Su, fu), d(Sx_{2n}, gx_{2n}),$$

$$\frac{1}{2} [d(Su, gx_{2n}) + d(Sx_{2n}, fu)]\})$$

$$- \varphi(\max\{d(fu, gx_{2n}), d(Su, fu), d(Sx_{2n}, gx_{2n})\})$$

$$\leq \psi(\max\{d(z, y_{2n-1}), d(Su, z), d(y_{2n}, y_{2n-1}),$$

$$\frac{1}{2} [d(Su, y_{2n-1}) + d(y_{2n}, z)]\})$$

$$- \varphi(\max\{d(z, y_{2n-1}), d(Su, z), d(y_{2n}, y_{2n-1})\})$$

$$\leq \psi(\max\{\frac{\epsilon}{2}, \epsilon, \frac{\epsilon}{2}, \frac{\delta}{2}\})$$

$$- \varphi(\max\{d(z, y_{2n-1}), \epsilon, d(y_{2n}, y_{2n-1})\}) \text{ for } n \geq N,$$

(since $\epsilon = d(Su, z)$)

$$= \psi(\epsilon) - \varphi(\max\{d(z, y_{2n-1}), \epsilon, d(y_{2n}, y_{2n-1})\})$$

$$\therefore \psi(d(Su, fx_{2n+1})) =$$

$$\psi(\epsilon) - \varphi(\max\{d(z, y_{2n-1}), \epsilon, d(y_{2n}, y_{2n-1})\})$$

On letting $n \rightarrow \infty$, we get,

$$\psi(\epsilon - 0) \leq \lim_{n \rightarrow \infty} \sup \psi(d(Su, fx_{2n+1})) \leq$$

$$\psi(\epsilon) - \varphi(\epsilon)$$

$$\leq \psi(\epsilon + 0) - \varphi(\epsilon)$$

$$\therefore \varphi(\epsilon) \leq \psi(\epsilon + 0) - \psi(\epsilon - 0), \text{ a contradiction.}$$

Hence $Su = z$.

Thus $Su = z = fu$ so that u is a coincidence point of (S, f) .

Now we prove that v is a coincidence point of (S, g) .

$$\begin{aligned} \psi(d(z, Sv)) &= \psi(d(Su, Sv)) \\ &\leq \psi(\max\{d(fu, gv), d(Su, fu), d(Sv, gv), \\ &\quad \frac{1}{2} [d(Su, gv) + d(Sv, fu)]\}) \\ &- \varphi(\max\{d(fu, gv), d(Su, fu), d(Sv, gv)\}) \\ &\leq \psi(\max\{d(z, z), d(z, z), d(Sv, z), \frac{1}{2} [d(z, z) + \\ &\quad d(Sv, z)]\}) - \varphi(\max\{d(z, z), d(z, z), d(Sv, z)\}) \\ &= \psi(d(Sv, z)) - \varphi(d(Sv, z)) \\ &\therefore \varphi(d(Sv, z)) = 0 \\ &\therefore d(Sv, z) = 0 \\ &\therefore Sv = z \\ &\therefore Sv = gv = z. \end{aligned}$$

Thus (S, f) and (S, g) have coincidence points, namely, u and v respectively.

Now suppose (S, f) and (S, g) are weakly compatible.

Then S and f commute at u and S and g commute at v .

Hence,

$$S(fu) = f(Su) \Rightarrow Sz = fz \text{ and}$$

$$S(gv) = g(Sv) \Rightarrow Sz = gz$$

$$\therefore Sz = fz = gz \rightarrow (2.4.15)$$

Now

$$\psi(d(z, Sz)) = \psi(d(Su, Sz)) \leq$$

$$\psi(\max\{d(fu, gz), d(Su, fu), d(Sz, gz), \frac{1}{2} [d(Su, gz) + d(Sz, fu)]\}) - \varphi(\max\{d(fu, gz), d(Su, fu), d(Sz, gz)\})$$

$$\leq \psi(\max\{d(z, Sz), d(z, z), d(gz, gz), \frac{1}{2} [d(z, Sz) + d(Sz, z)]\}) - \varphi(\max\{d(z, Sz), d(z, z), d(Sz, Sz)\})$$

$$= \psi(d(z, Sz)) - \varphi(d(z, Sz))$$

$$\therefore \varphi(d(z, Sz)) = 0$$

$$\therefore d(z, Sz) = 0$$

$$\therefore Sz = z$$

So that z is a fixed point of S .

Now, from (2.4.15) follows that z is a common fixed point of S, f and g .

Now suppose p is a common fixed point of S, f and g .

Then,

$$\psi(d(z, p)) = \psi(d(Sz, Sp)) \leq$$

$$\psi(\max\{d(fz, gp), d(Sz, fz), d(Sp, gp), \frac{1}{2} [d(Sz, gp) + d(Sp, fz)]\}) - \varphi(\max\{d(fz, gp), d(Sz, fz), d(Sp, gp)\})$$

$$\leq \psi(\max\{d(z, p), d(z, z), d(p, p), \frac{1}{2} [d(z, p) + d(p, z)]\}) - \varphi(\max\{d(z, p), d(z, z), d(p, p)\})$$

$$= \psi(d(z, p)) - \varphi(d(z, p))$$

$$\therefore \psi(d(z, p)) \leq \psi(d(z, p)) - \varphi(d(z, p))$$

$$\therefore \varphi(d(z, p)) = 0$$

$$\therefore d(z, p) = 0$$

$$\therefore z = p$$

$\therefore z$ is the unique common fixed point of S, f and g

The following is a direct corollary to our main result.

Corollary 2.5:

Let (X, d) be a complete metric space. Let $S, f: X \rightarrow X$ be self mappings such that for all $x, y \in X$, $S(X) \subset f(X)$,

$$\psi(d(Sx, Sy)) \leq \psi(\max\{d(fx, fy), d(Sx, fx), d(Sy, fy), \frac{1}{2}[d(Sx, fy) + d(Sy, fx)]\}) - \varphi(\max\{d(fx, fy), d(Sx, fx), d(Sy, fy)\}) \rightarrow (2.5.1)$$

where,

(2.5.2): $\varphi: [0, \infty) \rightarrow [0, \infty)$ is a continuous function with $\varphi(t) = 0$ if and only if $t = 0$.

(2.5.3): $\psi: [0, \infty) \rightarrow [0, \infty)$ is a pseudo altering distance function, and

$$(2.5.4): \psi(\epsilon + 0) - \psi(\epsilon - 0) < \varphi(\epsilon) \forall \epsilon > 0.$$

Then S and f have a coincidence point. Further if (S, f) is a weakly compatible pair, then S and f have a unique common fixed point.

Proof:

By taking $f = g$ in the theorem (2.4) we get the proof.

Now we show that the result of Gairola & Ram Krishan (2013) follows as a corollary from our main result.

Corollary 2.6: [Gairola & Ram Krishan (2013), Theorem 3.1]

Let (X, d) be a complete metric space. Let $S, f, g: X \rightarrow X$ be self mappings such that for all $x, y \in X$, $S(X) \subset f(X) \cap g(X)$,

$$\psi(d(Sx, Sy)) \leq \psi(\max\{d(fx, gy), d(Sx, fx), d(Sy, gy), \frac{1}{2}[d(Sx, gy) + d(Sy, fx)]\}) - \varphi(\max\{d(fx, gy), d(Sx, fx), d(Sy, gy)\}) \rightarrow (2.6.1)$$

where,

$\varphi: [0, \infty) \rightarrow [0, \infty)$ is a continuous function with $\varphi(t) = 0$ if and only if $t = 0$, and $\psi: [0, \infty) \rightarrow [0, \infty)$ is an altering distance function.

Then (S, f) and (S, g) have a coincidence point. Further if (S, f) and (S, g) are weakly compatible pairs, then S, f and g have a unique common fixed point.

Proof:

We have already observed (Remark following definition 2.1) that every altering distance function is a pseudo altering distance function. Further, if ψ is an altering distance function and φ is as stated in the above theorem, then clearly, for any $\epsilon > 0$,

$$\psi(\epsilon + 0) - \psi(\epsilon - 0) = 0 \text{ (Since } \psi \text{ is continuous at } \epsilon) < \varphi(\epsilon) \text{ (Since } \epsilon > 0)$$

Thus condition (2.4.4) in our main result (theorem 2.4) is satisfied.

Thus, all the hypothesis in theorem 2.4 is satisfied. Consequently, the result follows.

IV. CONCLUSIONS

The main result shows that existence of fixed points for self maps can be obtained even with the use of altering distance

functions which is more convenient than expecting the altering distance function to be continuous. Further finding a link between the pseudo altering distance function ψ and deficit function φ , which is automatically satisfied with an altering distance function, is a major positive step in improving common fixed point results.

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REFERENCES

- [1] Ya I. Alber & S.D. Guerre (1997), "Principle of Weakly Contractive Maps in Hilbert Space", *New Results in Operation Theory*, Editors: I. Goldberg, Yu, Lyabich, *Advance and Appl.*, Vol. 98, Birkhaser Verlag, Pp. 7-22.
- [2] P.K. Choudhary, B.E. Rhoades & N. Metiya (2011), "Fixed Point Theorem for Generalized Weakly Contractive Mappings", *Non-linear Analysis*, Vol. 74, Pp. 2116-2126.
- [3] R. Chugh & S. Kumar (2001), "Common Fixed Point for Weakly Compatible Maps". *Pro.Indian.Acad.Sci.*, Vol. 111, No. 2, Pp. 241-247.
- [4] U.C. Gairola & Ram Krishan (2013), "Common Fixed Point Theorem for Three Maps by Altering Distances between the Points", *IJMA*, Vol. 4, No. 1, Pp. 143-151.
- [5] G. Jungck (1986), "Compatible Mapping and Common Fixed Points", *Int.Indian Math and Math.Sci.*, Vol. 9, Pp. 771-779.
- [6] G. Jungck & B.E. Rhoades (1998), "Fixed Point for Set Valued Function without Continuity", *International Journal of Pure and Applied Mathematics*, Vol. 29, No. 3, Pp. 381-386.
- [7] M.S. Khan, M. Swaleh & S. Sessa (1984), "Fixed Point Theorem by Altering Distance between the Points", *Bulletin - Australian Mathematical Society*, Vol. 30, Pp. 1-9.
- [8] S.V.R. Naidu (2003), "Some Fixed Point Theorems in Metric Spaces by Altering Distances", *Czechoslovak Mathematical Journal*, Vol. 53, No. 1, Pp. 205-212.
- [9] O. Popescu (2011), "Fixed Point for (ψ, φ) - Weak Contractions", *Applied Mathematics Letters*, Vol. 24, Pp. 1-4.
- [10] B.E. Rhoades (1977), "A Comparison of Various Definitions of Contractive Mappings", *Transactions of the American Mathematical Society*, Pp. 257-290.
- [11] B.E. Rhoades (2001), "Some Theorems on Weakly Contractive Maps", *Non-linear.Anal.TMA*, Vol. 47, No. 4, Pp. 2683-2693.
- [12] K.P.R. Sastry & G.V.R. Babu (1999), "Some Fixed Point Theorems by Altering Distances between the Points", *Indian Journal of Pure and Applied Mathematics*, Vol. 30, No. 6, Pp. 641-647.
- [13] A. Singh & R.C. Dimri (2011), "A Common Fixed Point Theorem through Generalized Altering Distance Functions", *Kochi Journal of Mathematics*, Vol. 6, Pp. 149-157.