

A Delayed Pulse Vaccination Model of Infectious Diseases in Children

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Abstract—This study concerns the theoretical determination of a mathematical model of delayed pulse vaccination of infectious diseases that affects children. In this study, a delayed SEIR epidemic model with impulsive effect and the global dynamic behaviors of the model will be analyzed. Using the discrete dynamical systems determined, it's shown that there exists an 'infection-free' periodic solution which is globally attractive when the period of impulsive effect is less than certain critical value. The sufficient condition for the permanence of the epidemic model with pulse vaccination is given, which means the epidemic disease is to spread around. The study has concluded that time delay and pulse vaccination brings great effects of shortening 'infection period' on the dynamics of the model. The results indicate that a large vaccination rate or a short period of pulsing leads to the eradication of the disease. Numerical simulation has been used together with the analytical results. The result has been presented in tabular and graphical form.

Keywords—Basic Reproduction Ratio; Compartmental Model; Disease-Free Equilibrium; Infectious Diseases; Pulse Vaccination; Time Delay.

Abbreviations—Delay Differential Equation (DDE); Ordinary Differential Equation (ODE); Pulse Vaccination Strategy (PVS); Susceptible, Exposed, Infectious and Recovered (SEIR).

I. INTRODUCTION

INFECTIONOUS diseases are disorders caused by pathogenic microorganisms. Most infectious diseases could be driven towards eradication, if adequate and timely steps (e.g. vaccination, treatment, etc.) are taken in the course of an epidemic. However, many of these diseases eventually become endemic in societies due to lack of adequate policies and timely interventions to mitigate the spread of the diseases. Consequently, there is the need for proactive steps towards controlling the spread of infectious diseases, particularly those ones for which both vaccine and cure are available.

The ultimate goal of an epidemic model would be to closely follow and predict real-life disease outbreaks, with the aim of informing public policy and related government agencies. It will focus on looking at control methods, i.e. ways to keep the infective population low or to eradicate the infection altogether. One such control method is vaccination. Some vaccination campaigns are run continuously, for example with people of a certain age receiving their vaccine. Another way is to organize large campaigns in which a large

proportion of the population is vaccinated over a short time; this technique is known as pulse vaccination.

In modeling of delayed pulse vaccination of infectious diseases, the study focuses on deterministic models and dynamical systems used to model epidemics, using deterministic compartmental models, in which a given population is divided into compartments based on the disease status (susceptible, exposed etc.). The transfers between compartments, as well as the entrance to the population of new individuals and the exit of others are modeled as terms in a differential equation governing the time-evolution of each compartmental value. After the infectious individuals lived through an infection period, they recover completely and transfer to the 'removed' class, R, so, the number of the death of the infectious should be considered during convalescence, which is called the phenomena of 'time delay'.

Therefore, this study will consider an epidemic model with impulsive vaccination and time delay and study their dynamic behaviors under pulse vaccination. The main aim of this study is to introduce time delay, pulse vaccination in an epidemic model and to obtain some important qualitative properties and valid pulse vaccination strategy.

II. RELATED WORKS/LITERATURE SURVEY

WencaiZhao et al.,[7] researched on dynamical analysis of SIR epidemic model with non-linear pulse vaccination and lifelong immunity. In their study, due to the limited medical resources, vaccine immunization rate is considered as a nonlinear saturation function and their findings were enriching medical resources the disease will be in extinction, otherwise the disease will be permanent.

Onyejekwe&Kebede[14] studied the epidemiological modeling of measles infection with optimal control of vaccination and supportive treatment, in which they concluded that the optimal combination of the strategies required to achieve the set objective depend on the relative cost of each of the control measures and the resulting optimality system. The use of both vaccination and supportive treatment gives the highest possible rate to the control of epidemics.

Tongqian et al.,[10], in their study SVEIRS a new epidemic disease model with time delays and impulsive effects realized that global dynamical behavior of the model with pulse vaccination and impulsive population inputs effects at two different periodic moments, existence and global attractivity of the infection free periodic solution and also permanence of the model. Their results shows that time delay, pulse vaccination and pulse population input can exert a significant influence on the dynamics of the systems which confirms the availability of pulse vaccination strategy for the practical epidemic prevention.

Shulgin et al.,[16] considered a simple SIR model with pulse vaccination and have shown that if certain conditions regarding the magnitude of vaccination proportion and on the period of pulses are satisfied then the pulse vaccination leads to epidemic eradication.

Rui&Yanke[15] investigated a delayed SIR epidemic model with nonlinear incidence rate and pulse vaccination, they noted that the global attractiveness of infection free periodic solution was analyzed and sufficient conditions are obtained for permanence of the system. Their results indicated that a large vaccination rate or a short period of pulsing leads to the eradication of the disease.

An epidemic HIV/AIDS model with treatment has been investigated in the study by Shaoying et al., [11]. The model allows some infected individuals to move from symptomatic phase to the asymptomatic phase by all kinds of treatments. The authors introduced the time delay to the model in order to investigate the effect of the time delay on the stability of the endemically infected equilibrium. This discrete time delay has also been used to the model to describe the time from the start of the treatment in the symptomatic stage until the treatment effects becomes clear. It was found that treatment can be used to make the disease free equilibrium (E_0) stable when it would be unstable in the absence of treatment. On the other hand using the time delay can induce oscillation in the system. Biologically, this means that there is a critical value for the treatment-induced delay which determines the

stability of the infected equilibrium E^* . That is, the infected equilibrium E^* is asymptotically stable when antiretroviral drugs on average show positive effects in patients within less than time delay.

The control of epidemics by vaccination, by Verriest et al.,[17], they used recently developed results on optimal impulsive control for time delay systems in the problem of control of an epidemic through pulse vaccination. For added realism, delays are explicitly incorporated in the epidemiological model. It was shown that the conditions for optimality are easily amenable by an iterative gradient type numerical algorithm. They recommended future work to include multipulse strategies. They expected that current policies of periodic vaccination pulses can be improved upon. This will then provide a 'proof of principle' with which more realistic models for disease may be attacked.

The combination of pulse vaccination in an epidemic model with time delay is the main objective of this study, focusing on pulse vaccination. The study of the pulse vaccination model with delay as given by Gao et al., [6] will be the basis of this research.

2.1. Purpose of the Study

Infectious diseases have been a major concern in health sector, as it affects children and young borns adversely. Constant vaccination have been used mostly as a method of controlling infectious diseases e.g. measles, polio, etc. Pulse vaccination is the latest advancement in health sectors hence its study.

2.2. Objectives of the Study

The main objective of this study is to model the infectious diseases, come up with the control measures to enable their eradication and determine the effect of the various population parameters on the delayed pulse vaccination using delayed differential equations, also to:

- i. To formulate a mathematical model for infectious diseases with or without pulse vaccination.
- ii. To determine the local and global stability of the equilibrium points.
- iii. To analyze the model qualitatively.

III. METHODS/DISCUSSION

In this study we formulate and analyse the deterministic compartmental model of the infectious disease on the population. A deterministic compartmental model is one in which the individuals in a population are classified into compartments depending on their status with regard to the infection, the compartments are; Susceptible – $S(t)$, Exposed/latent but not yet infectious – $E(t)$, Infected – $I(t)$ and the Removed – $R(t)$ for $SEIR$ model. The differential equations (DE) derived from the model will be assigned initial conditions (IC) and boundary conditions (BC) which will help to solve them. The various parameters will be determined for the dynamical system. The differential equations governing the deterministic models have been used.

In mid-19th century, the Xinzhi&Peter [13] discussed different theorems of aiding in solving partial differential equations.

The existence and uniqueness theorem, the stability theorems, and the Comparison theorems are applicable to the general systems of differential equations, based on the equations of D'Onofrio[2].

a) Non-delay SIR Model

$$\left. \begin{aligned} \frac{dS}{dt} &= b(N(t) - S(t)) - \beta \frac{I(t)}{N(t)} S(t) \\ \frac{dI}{dt} &= \beta \frac{I(t)}{N(t)} S(t) - (\mu + \gamma) I(t) \\ \frac{dR}{dt} &= \gamma I(t) - \mu R(t) \end{aligned} \right\} \quad (1)$$

b) Delay SEIR Model

$$\left. \begin{aligned} \frac{dS}{dt} &= b(N(t) - S(t)) - \beta \frac{I(t)}{N(t)} S(t) \\ \frac{dE}{dt} &= \beta \frac{I(t)}{N(t)} S(t) - \beta e^{-\mu r} \frac{I(t-r)}{N(t-r)} S(t-r) - \mu E(t) \\ \frac{dI}{dt} &= \beta e^{-\mu r} \frac{I(t-r)}{N(t-r)} S(t-r) - (\mu + \gamma) I(t) \\ \frac{dR}{dt} &= \gamma I(t) - \mu R(t) \end{aligned} \right\} \quad (2)$$

c) The DELAY SEIR model with Pulse vaccination

$$\left. \begin{aligned} \frac{dS}{dt} &= bN(t) - \mu S(t) - \beta \frac{I(t)}{N(t)} S(t) \\ \frac{dE}{dt} &= \beta \frac{I(t)}{N(t)} S(t) - \beta e^{-\mu r} \frac{I(t-r)}{N(t-r)} S(t-r) - \mu E(t) \\ \frac{dI}{dt} &= \beta e^{-\mu r} \frac{I(t-r)}{N(t-r)} S(t-r) - (\mu + \gamma) I \\ \frac{dR}{dt} &= \gamma I - \mu R \\ S(t) &= (1-p)S(t^-) \\ E(t) &= E(t^-) \\ I(t) &= I(t^-) \\ R(t) &= R(t^-) + pS(t^-) \end{aligned} \right\} \quad \begin{aligned} &t \neq \kappa\tau \\ &t = \kappa\tau \end{aligned} \quad (3)$$

3.1. Ordinary Differential Equations

An ordinary differential equation (ODE) is an equation that involves some ordinary derivative of a function which can be solved by integration.

$$x'(t) = f(t, x) \quad (4)$$

This equation is non-autonomous since it depends explicitly on the time variable t in addition to the state variable $x(t)$. It's assumed that the ODE is subject to the initial condition (IC).

$$x(t_0) = x_0 \quad (5)$$

These theorems are:

The Existence and uniqueness theorem: Local existence theorem

Theorem 1: Peano's Existence Theorem: Let $f \in C(F, R^n)$, that is, f is a continuous function from F to R^n where

$$F = \{(t, x) \in R \times R^n : |t - t_0| \leq a, \|x - x_0\| \leq c, a, c > 0\} \quad (6)$$

and let, $\|f(t, x)\| \leq M$ on F for some $M > 0$. Then the IVP (4-5) has at least one solution $x(t)$ defined on $[t_0 - \alpha, t_0 + \alpha]$ where $\alpha = \min(a, \frac{c}{M})$.

If $b = \mu$ and the population is normalized to $N(t) = S(t) + I(t) + R(t) \equiv 1$, equation (1) becomes:

$$\left. \begin{aligned} \frac{dS}{dt} &= \mu(1 - S(t)) - \beta I(t)S(t) \\ \frac{dI}{dt} &= \beta I(t)S(t) - (\mu + \gamma)I(t) \\ \frac{dR}{dt} &= \gamma I(t) - \mu R(t) \end{aligned} \right\} \quad (7)$$

Defining $x = [S, I, R]^T$ and $f(t, x) := [x^1, x^2, x^3]^T$. Then (1) is equivalent to $x'(t) = f(t, x)$.

In this normalized case, the physical region is $x \in \Omega_1 := \{(S, I, R) \in [0, 1]^3 : S + I + R = 1\}$ since S, I , and R , are fractions of the population. This region is positively invariant.

If the birth and death rates are unequal, $b \neq \mu$, then the boundedness of the model is complex, since the population sizes may grow. The physical region of interest is now (potentially) unbounded; thus we define $\Omega_N := R_+^3$ (where $R^+ = [0, \infty)$). The region Ω_N is positively invariant with respect to the DE 4:

$$S(t) = 0 \Rightarrow S' = \mu > 0; I(t) = 0 \Rightarrow I' = 0; R(t) = 0 \Rightarrow R' = \gamma I(t) \geq 0 \quad (8)$$

So with these initial conditions in Ω_N , the trajectory of the solution will never leave Ω_N . This analysis assumes the total population will undergo exponential growth or decay depending on the relative values of b and μ .

IV. SEIR MODEL WITHOUT VACCINATION

The differential equations for this model is system 2 when $r = 0$ then the system becomes;

$$\left. \begin{aligned} \frac{dS}{dt} &= b(N(t) - S(t)) - \beta \frac{I(t)}{N(t)} S(t) \\ \frac{dE}{dt} &= \beta S(t) \frac{I}{N} - (\sigma + \mu)E(t) \\ \frac{dI}{dt} &= \sigma E(t) - (\gamma + \mu + \delta)I(t) \\ \frac{dR}{dt} &= \gamma I(t) - \mu R(t) \end{aligned} \right\} \quad (9)$$

$$\frac{dN}{dt} = 0, \text{ and } N = S + E + I + R \text{ is thus constant.}$$

4.1. Properties of the SEIR Model Equations

The basic properties of the of the model equations 9 are feasible solutions and positivity of solutions.

A. Feasible Solution

The feasible solution set which is positively invariant set of the model is given by,

$$\emptyset = \{(S, E, I, R) \in R : S + E + I + R = N \leq \frac{b}{\mu}\} R_+^4$$

B. Positivity of Solutions

A first-order linear differential equation of the form, $\frac{dN}{dt} = (b - \mu)N$. Thus $N(t) = Ce^{(b-\mu)t}$ at $t = 0, N(0) = C$

Hence the solution of the differential equation then becomes $N(t) = N(0)e^{(b-\mu)t}$ therefore, $\emptyset$ is positively invariant.

C. Existence of Steady States of the System

The equilibrium points of the system can be obtained by equating the rate of changes to zero, $\frac{dS}{dt} + \frac{dE}{dt} + \frac{dI}{dt} + \frac{dR}{dt} = 0$.

D. Global Asymptotic Stability of the Model

In proving the global stability of the SEIR Model, there is need to find the equilibrium points of the system

Assuming that the birth rate, b is equal to death rate, μ i.e. $b=\mu$.

V. NUMERICAL SIMULATION

5.1. The Analysis of the SEIR Model without Pulse Vaccination

This section gives an illustration of the analytical results of the SEIR model without pulse vaccination by carrying out stability analysis and numerical simulations of the model using the parameter values pertinent to Kenya given in Table 1 below. These parameters were obtained from different sources in the literature.

Table 1: Parameter Symbol, Values and their Sources

Parameter Symbol	Parameter Value	Literature Source
b	0.02755 per year	Gao et al., [5]
μ	0.00875 per year	Gao et al., [6]
β	0.09091 per day	D'onofo[1]
σ	0.125 per day	Gao et al., [4]
α	0.14286 per day	Momoh et al., (2013)

5.2. Stability Analysis of the Model

From model equation (3) when no time delay then the equations becomes equation (10).

$$\begin{aligned}\frac{dS}{dt} &= b(N(t) - S(t)) - \beta \frac{I(t)}{N(t)} S(t) \\ \frac{dI}{dt} &= \beta \frac{I(t)}{N(t)} S(t) - (\mu + \gamma) I(t) \\ \frac{dR}{dt} &= \gamma I(t) - \mu R(t)\end{aligned}\quad (10)$$

Endemic Model

$$\begin{aligned}\frac{ds}{dt} &= \mu - (\mu + \beta i)s \\ \frac{de}{dt} &= \beta si - (\mu + \sigma)e \\ \frac{di}{dt} &= \beta si - (\mu + \sigma)e \\ \frac{dr}{dt} &= (\gamma + \delta)i - \mu r\end{aligned}\quad (11)$$

Linearising the system of the differential equations, the Jacobian matrix is given as

$$J(s, e, i, r) = \begin{bmatrix} \mu + \beta i & \mu & 0 & \mu \beta \\ \beta i & \mu + \sigma & \mu & b \\ 0 & b & \mu + \gamma + \delta & 0 \\ 0 & 0 & 0 & \gamma + \delta \end{bmatrix}\quad (12)$$

For the infection free equilibrium $(s, e, i) = (1, 0, 0)$, the Jacobian matrix then becomes

$$J(1, 0, 0) = \begin{bmatrix} -\mu & 0 & -\beta \\ 0 & -(\mu + \delta) & \beta \\ 0 & \delta & -(\mu + \sigma) \\ -0.0875 & 0 & -0.09091 \\ 0 & -0.2125 & 0.09091 \\ 0 & 0.125 & -0.230336 \end{bmatrix} = \quad (13)$$

The important sub-matrix is the second 2x2 matrix. From this, the trace $(T) < 0$, but if $R_0 < 1$, then the determinant $(D) > 0$ and if $R_0 > 1$ then $(D) < 0$ for all parameters, Routh-Hurwitz stability condition for T and D is given as follows: $T = -(2\mu + \sigma + \alpha) = -0.44286$ and $D = (\mu + \alpha)(\mu + \sigma)(1 - R_0) = 0.02488$ Hence the disease free study state when $R_0 < 0$, and unstable when $R_0 > 0$. The eigenvalues at the disease free equilibrium are given by $\{-\mu, -(\mu + \sigma), -(\mu + \alpha)\}$. All the eigenvalues are negative meaning the disease free equilibrium $(1, 0, 0, 0)$ is asymptotically stable. The endemic equilibrium $(s^*, e^*, i^*) = \left(\frac{1}{R_0}, \frac{\mu(R_0 - 1)}{R_0(\mu + \sigma)}, \frac{\mu(R_0 - 1)}{\beta}\right)$

The Jacobian matrix for the endemic equilibrium is given as

$$J_{\text{endemic}} = \begin{bmatrix} -\mu R_0 & 0 & -(\mu + \alpha)(\mu + \sigma) \\ \mu(R_0 - 1) & -(\mu + \sigma) & (\mu + \alpha)(\mu + \sigma) \\ 0 & \sigma & -(\mu + \alpha) \end{bmatrix}\quad (14)$$

Whose Characteristic equation is given as $X^3 + a_1 X^2 + a_2 X + a_3 = 0$, Routh-Hurwitz criteria for stability, all the roots of the Characteristic equation have negative real part which means stable equilibrium is attained.

5.3. Optimal Vaccination Strategies: Herd Immunity

Herd immunity is the level of immunity in a population which prevents epidemics, even if some transmission may still occur. It is well-known that the higher R_0 is for a disease, the higher the proportion of the population will have to be vaccinated to achieve herd immunity as seen by Hethcote[8]. Although, this statement could seem theoretical, it was almost the perspective followed by WHO's Technical Working Group (2000), when devising strategies to control a full range of diseases; for instance, this procedure has succeeded during the worldwide campaign for measles and smallpox eradication in the 1960s. Let p be the proportion immune after a vaccination campaign. To reach the so-called critical proportion p_c , we need the control condition $R_0(1 - p_c) < 1$ to be fulfilled. For instance, in most sub-Saharan Africa countries, the basic reproductive number for measles R_0 is approximately around 18 by Hethcote[12] and Grais (2006), so $p_c = 0.94$. Under the schedule of an unique dose, the minimal coverage to control infectious diseases is such that everyone does not need be immune through vaccination to control infectious diseases.

5.4. Numerical Simulations and Analysis of the Simulations of the SEIR Model Equations

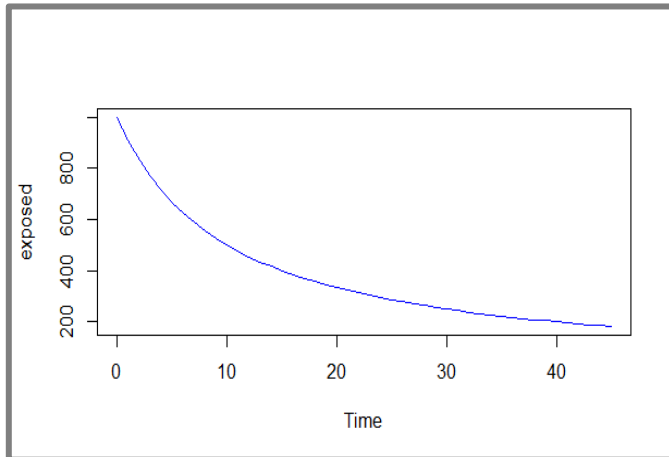


Figure 1: Simulation of the Exposed Population

In Figure 1 It can be observed that as the rate increases, the population of exposed individuals shows some rapid decrease. The decrease in the exposed population could be due to early detection, pulse vaccination and also possibly due to those who enter the infective class. This decrease could also be due to the education about the infectious diseases transmission, very few individuals are coming out as infected individuals. Also the dynamics of the exposed population depend on the contact number.

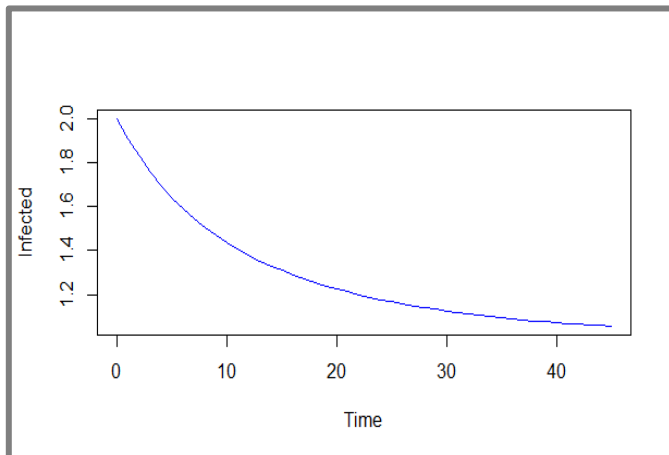


Figure 2: Simulation of the Infected Population

In Figure 2, it is realized that the population of infected individuals at the very beginning fall sharply as the rate increases and then fall uniformly as time increases. This rapid decline of the infected individuals may be due to early detection of the measles and partly due to those who revert to the Exposed class. This graph also demonstrates that the contact rate has large impact on the spread of the disease through population. If the contact rate is observed to be high then the rate of infection of the disease will also be high as would be expected logically. However, there exists another parameter to consider as more individuals are infected with the disease and $I(t)$ grows, as some individuals are leaving the infected class by being cured and joining the recovered class.

VI. DISCUSSION

The main objective of this study is to model delayed pulse vaccination of infectious diseases and also establish a possible way of reducing the disease transmission.

The basic reproductive number has been computed to determine the stability of the disease because theoretical determination of threshold conditions for R_0 is of important public health interest. It was established qualitatively that the model has the disease-free equilibrium and endemic equilibrium points. It was realized that whenever $R_0 < 1$, the disease-free equilibrium point is locally asymptotically stable and unstable whenever $R_0 > 1$.

It was also realized that, in the absence of mass vaccination programme as well as early detection and supervised treatment, the transmission of the disease cannot be eradicated from the population. The introduction of proper treatment and education about the disease transmission as well as early detection of the disease can help reduce the disease in a population. The results has also shown that effective contact with the infectious individual cause a major increase of the disease transmission, hence individuals with active infectious disease must be detected as early as possible to reduce high rate of transmission in a population. Education about infectious disease can help many appear for diagnosis and get detected.

6.1. Model Analysis: The SIR Model with Vital Dynamics

In the SIR Model with vital dynamics, an SIR epidemiological Model is considered, but here a model of the disease behavior in the population over a long time period. A disease is called endemic if it is present in a population for more than 10 or 20 years. Because of the long period involved, a model for an endemic disease must include births as a source of new susceptible and natural deaths in each class.

From the above discussions, we know that the set Ω_0 is a global attractor in Ω , and of course, every solution of system 4 with initial conditions 5 will eventually enter and remain in region Ω_0 . Therefore, system 4 is permanent. Solving system 9-12 numerically we get the diseased states as:

$$S^* = \frac{(\sigma + \alpha + \mu)(\gamma + \mu + \delta)}{\alpha\beta} \quad (15)$$

$$E^* = \frac{\alpha\beta B(\varphi + \mu) + \phi\lambda(\sigma + \alpha + \mu)(\gamma + \mu + \delta)}{\beta(\varphi + \mu)(\sigma + \alpha + \mu) - \left(\frac{\lambda + \mu(\gamma + \mu + \delta)}{\alpha\beta}\right)} \quad (16)$$

$$E^* = \frac{\sigma\alpha\beta}{\mu(\sigma + \alpha + \mu)(\gamma + \mu + \delta)} + \frac{\lambda\sigma\varphi(\gamma + \mu + \delta)}{\mu\beta(\varphi + \mu)} - (\lambda + \mu)\frac{(\gamma + \mu + \delta)}{\alpha\beta\mu} + \frac{\alpha\gamma\beta}{\mu(\sigma + \alpha + \mu)} \quad (17)$$

$$I^* = \frac{\alpha\beta B(\varphi + \mu) + \phi\lambda(\sigma + \alpha + \mu)(\gamma + \mu + \delta)}{\beta(\varphi + \mu)(\sigma + \alpha + \mu)(\gamma + \mu + \delta) - \left(\frac{\lambda + \mu}{\beta}\right)} \quad (18)$$

$$R^* = \frac{(\sigma + \alpha + \mu)(\gamma + \mu + \delta)}{\alpha\beta(\varphi + \mu)} \quad (19)$$

We have analyzed the SIR epidemic model with pulse vaccination and time delay. Two thresholds have been established, one for global stability of the infectious-free solution and one for persistence of the endemic solution.

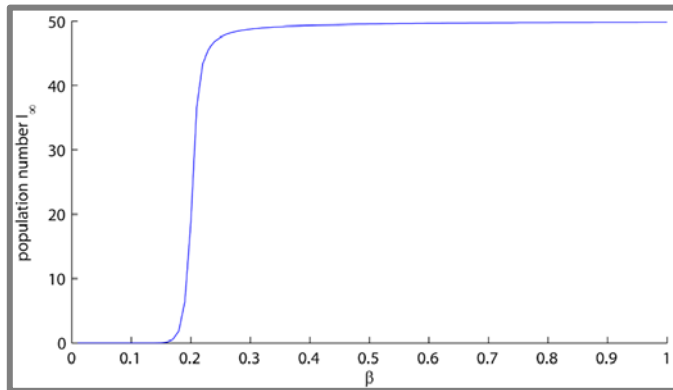


Figure 3: The figure of the unique endemic equilibrium (the component I of infectious individuals regarding β as the contact rate parameter, all other parameters are the same)

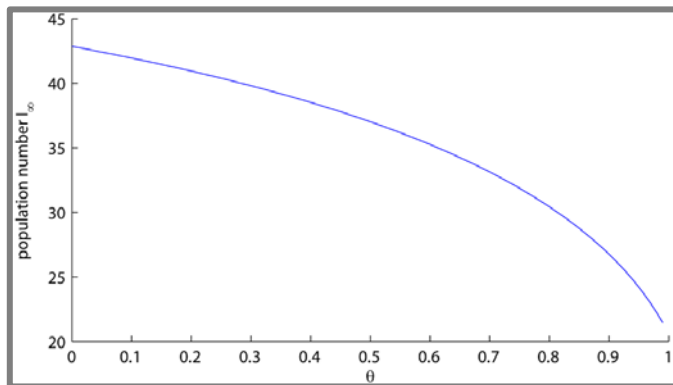


Figure 4: The figure of the unique endemic equilibrium (the component I of infectious individuals regarding θ as the pulse vaccination parameter, all other parameters are the same) except for $\beta = 0.214$

From Figures 3 and 4, we can observe the following:

(i) R^* and R_* are inversely proportional to p value and directly proportional to τ value and R_0 value, which implies that pulse vaccination p measures the inhibition effect from the behavioral change of the susceptible when they transfer to the infectious class (I).

(ii) R^* is a directly proportional to μ value, which implies that the natural birth or death rate measures the inhibition effect from the behavioral change of the susceptible class (with S) when it moves into the infectious class (I).

(iii) R_* is inversely proportional to h value, which implies that the maximum infectious period of the disease measures the inhibition effect from the behavioral change of the susceptible class (with S) when it moves into the infectious class ($I(t)$).

(iv) There is a value μ^* such that R_* is directly proportional to μ when $\mu < \mu^*$ and is inversely proportional to μ when $\mu > \mu^*$. Therefore the larger death rate is sufficient for the global attractivity of infection free periodic solution

($\tilde{S}e(t), 0$). It is easy to verify. In fact, we can calculate the derivative of R_* with respect to μ

$$\frac{dR_*}{d\mu} = \frac{(1-p)e^{-\mu h} R_0}{[1 - (1-p)e^{-\mu h}]^2} g(\mu) \quad (20)$$

Where $g(\mu) = p\tau e^{-\mu\tau} - h(1-e^{-\mu\tau})(1 - (1-p)e^{-\mu\tau})$. Obviously, $g'(\mu) < 0$ and $g(0) > 0$, $\lim_{\mu \rightarrow +\infty} g(\mu) < 0$. Hence, there exists a μ^* such that $dR_*/d\mu > 0$ for $\mu \in (0, \mu^*)$, whereas $dR_*/d\mu < 0$ for $\mu \in (\mu^*, +\infty)$.

Epidemic models with time delays have received much attention since delays can often cause some complicated dynamical behaviors. Delays in many models can destabilize equilibrium and thus lead to periodic solutions by Hopf bifurcation Hethcote[12], Cooke & Busenberg[9]. It is well known that periodic forcing can drive SIR and SEIR models into a behavior which looks chaotic.

The impulsive model with time delay (3) will be analyzed, in particular paying attention to the following points:

- The global asymptotic stability for SIR model with pulse vaccination and distributed time delay;
- The behavior of the model when an insufficient level of people undergoes the vaccination: bifurcation and chaotic solutions;
- Whether periodic or pulse vaccination does a better job than constant vaccination at the same average value.

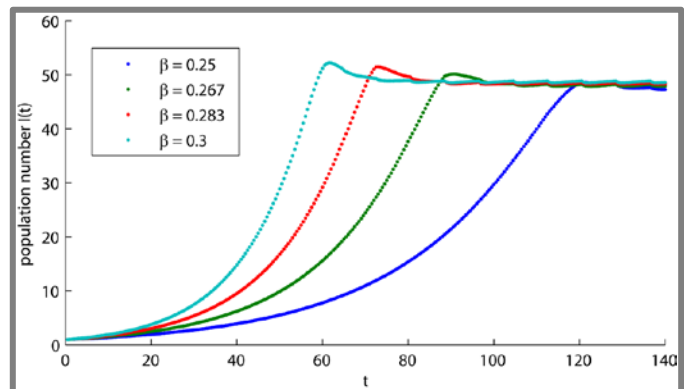


Figure 5: The tendency of the infected individuals I with different values of β

Therefore, an interesting open problem is proposed whether we can prove that the positive periodic solution of model (3) is globally attractive as $R^* > 1$.

Finally, the numerical simulations of the stroboscopic map of model on the number of infected individuals with different values of β are shown in Figure 5. It shows that the number of infected individuals will increase steadily in next few days, then reach the peak and begin a slow decline, and finally become stable. The greater the value β , the bigger the peak value and the earlier the peak appears. Our result implies that decreasing infection rate can put off the disease outbreak and reduce the number of infected individuals.

ACKNOWLEDGEMENT

I would like to thank the Almighty God for his great love, good health and care He has given me in life and especially through this study. I would like to thank my supervisors Prof Johana Kibet Sigey (JKUAT) and Prof Jeconiah Abonyo Okello (JKUAT).

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